Multiple light scattering in a two-dimensional medium with large scatterers

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A study is presented of the two-dimensional case of light propagation in a turbid medium involving sharply anisotropic one-center scattering in accordance with the Heney–Greenstein law, with strong absorption (when the photon absorption length is shorter than the transport length). An exact analytic solution that takes into account fluctuations of the photon paths is obtained within the framework of the small-angle approximation for the transfer equation in the case of a light beam normally incident upon the plane surface of the medium. The parameters of the radiation distribution (angular spectrum and attenuation coefficient) for the deep-propagating mode are analyzed in detail.

1. INTRODUCTION

Studies of the propagation of radiation in various scattering media have recently aroused widespread interest.[1-5] Whereas two decades ago these problems excited mainly specialists in the fields of radio and optical communications and in the physics of the atmosphere and of the ocean, the recent interest in investigation of laws governing multiple scattering of light is related, to a large extent, to the fundamental issue of wave propagation in disordered media with various types of interaction.

One of the most important stages in the description of light propagation in a scattering medium consists in deriving the average intensity of the radiation from the solution of the transfer equation (Boltzmann's transport equation) or in the equation for the mutual coherence function that is related to the transfer equation by the Fourier transformation. Unfortunately, no exact analytical solution of the transfer equation in the general case can be obtained. Therefore the main efforts in this field are usually concentrated on the development of approximately analytical and numerical methods.

At the same time, of particular interest is the search for exactly solvable models that are useful for qualitative comprehension of the peculiarities of the propagation of radiation in disordered media. From this point of view it is necessary that we mention the exact solution of the problem in the case of isotropic scattering (the Wiener–Hopf method). A theory that also is undergoing rapid development is the small-angle theory of light propagation, which is valid for media with large-scale scattering centers ($a \gg \lambda$, where a represents the size of the scatterer and λ is the wavelength of light radiation divided by 2π). In this theory two approximations elaborated in detail are known: the standard small-angle approximation[2,5-11] and the approximation that takes into account the spreadout of the photon path lengths.[10,12-16] The approximate solu-

tion[13,14] is applied in the case of strong absorption ($l_a \ll l_{tr}$; l_a is the photon absorption length, and l_{tr} is the transport length of elastic scattering) and in the nonstationary problem.[15] However, this solution of the transfer equation, which takes into account the fluctuations of photon paths, can be obtained only in the diffusion approximation in the angular variable in the elastic collision integral. It is known that the diffusion approximation (the Fokker–Planck approximation) is applicable only if the phase function decreases, as the angle of single scattering increases, more rapidly than γ^{-4}, where γ is the single-scattering angle.[17,18] In practice this condition is usually not satisfied. Therefore the results[13-15] may serve only for qualitative analysis of the laws governing light propagation. In many natural and artificial media the phase functions decrease more slowly than γ^{-4}, and for studying the propagation of radiation in these media it is necessary that one make use of other approximate methods for solving the transfer equation.[16,18,19] However, since no exact solution of the problem in the small-angle approximation with account taken of absorption has been found yet for phase functions of the sort indicated, the precision of the above-mentioned approximate methods can be estimated only by comparison with experimental data or with the results of numerical calculations.

Hitherto in our discussion we have dealt with the scattering of light in an ordinary three-dimensional (3D) medium, and all assertions were relevant only to that case. Recently, however, a number of publications have raised the issues of multiple scattering of light in two-dimensional (2D) matter, which can be represented as an ensemble of disordered parallel fibers or rods.[20-25] The interest in 2D scattering is related to both analytical and numerical investigations of radiation transport in media with oriented, essentially nonspherical, scatterers[23-25] and to the problem of localization of waves in random media.[20-22,25] Experiments with 2D multiple scattering were

performed with optical fibers[20] and with anisotropic scatterers (e.g., rods[21]).

Unlike in the case of a 3D medium, in the 2D small-angle case there exist, besides the known small-angle-diffusion approach to the problem of light propagation, other exactly solvable models. One of the most interesting models is the scattering model that complies with the Heney–Greenstein law and that describes the diffraction cross section averaged over oscillations.

As in the usual 3D case, the degree of scattering anisotropy depends on the ratio of the size of the scattering center a (in the 2D case, the ratio of the size of the transverse cross section of the fiberlike inhomogeneity) and the wavelength of the incident radiation λ. The effective multiple-scattering angle γ_{ef} can be estimated as $\gamma_{ef} \sim \lambda/a$; and therefore, when $\lambda \ll a$, the case that we are interested in, sharply anisotropic small-angle scattering, occurs.

In this limit case of scattering through small angles, the Heney–Greenstein, HG, 2D phase function can be written in the form

$$\chi^{HG}(\gamma) \approx \frac{1}{\pi} \frac{\gamma_{ef}}{\gamma_{ef}^2 + \gamma^2}. \tag{1}$$

γ_{ef} may be estimated from comparison with the diffraction cross section:

$$\gamma_{ef} = \frac{\lambda}{a}. \tag{2}$$

Phase function (1) is a particular case ($\nu = 1/2$) of the more general scattering law in the form

$$\chi_\nu(\gamma) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu)} \frac{\gamma_{ef}^{2\nu}}{(\gamma_{ef}^2 + \gamma^2)^{\nu+1/2}}, \tag{3}$$

which is convenient for analytical approximation of the experimentally determined cross sections.

Since the phase function in the case of sharply anisotropic scattering ($\gamma_{ef} \ll 1$) decreases rapidly with the increase of γ, all the essential information that is relevant to the scattering depends on the behavior of the scattering in the region of relatively small angles ($\gamma \ll 1$). Therefore small-angle 2D phase functions (1) and (3) may be considered to be normalized by the condition

$$\int_{-\infty}^{+\infty} \chi(\gamma) d\gamma = 1. \tag{4}$$

The region of applicability of the small-angle approximation is determined in problems of transport theory by the absorption coefficient κ ($\kappa = l_a^{-1}$). Thus in the case of weak absorption ($l_a \gg l_{tr}$) the small-angle approximation is valid only in the restricted region of depths $z < l_{tr}$ ($l_{tr} = 1/\sigma_{tr}$, where $\sigma_{tr} = \sigma\langle 1 - \cos\gamma\rangle$ is the transport scattering coefficient and σ is the scattering coefficient). The quantity l_{tr} can be considered the isotropic radiation length for the case of purely elastic scattering.

In strongly absorbing media with large-scale scattering centers ($a \gg \lambda$), when

$$l_a \ll l_{tr}, \quad \text{i.e., } \kappa \gg \sigma_{tr}, \tag{5}$$

isotropization of the beam does not occur, since photons undergoing strong scattering and thus propagating along more significantly bent trajectories are absorbed before

they have time to arrive at the depth of isotropization $z \sim l_{tr}$. This circumstance results in light always being scattered through small angles in such media[13-15]:

$$\langle\theta^2\rangle_z < \langle\theta^2\rangle_\infty \ll 1. \tag{6}$$

Here $\langle\theta^2\rangle_z$ is the mean square of the multiple-scattering angle at a depth z, and $\langle\theta^2\rangle_\infty$ is its limit value. Condition (6) permits application of the small-angle approximation at any depth. This turns out to be essential in the search for the analytical solution of the transfer equation.

Below, the exact analytic solution of the small-angle transfer equation, written so as to take into account fluctuations of the photon paths, is obtained for the 2D Heney–Greenstein phase function [relation (1)]. A detailed analysis of the solution is performed for various optical characteristics of the medium.

The asymptotic (deep) mode of radiation propagation has been considered in the case of strong absorption [relation (5)]. The angular spectrum, the total flux, and other parameters of the distribution are found. Explicit forms are obtained of various approximate representations of the exact solution, which forms are useful both for a qualitative understanding of the laws of light propagation in a medium and for practical calculations.

2. FORMULATION OF THE PROBLEM

Consider a broad stationary light beam of intensity I_0, incident upon the plane boundary of a 2D medium, in the xOz plane in which scattering occurs and which occupies the semispace $z > 0$ (the z axis is normal to the surface and directed into the medium); see Fig. 1.

We assume single scattering to be sharply anisotropic ($1 - \langle\cos\gamma\rangle \ll 1$) and the condition of strong absorption ($l_a \ll l_{tr}$) to be satisfied.

The transfer equation for the radiation intensity $I(z,\theta)$ is of the following form[23]:

$$\cos\theta \frac{\partial I(z,\theta)}{\partial z} + \kappa I(z,\theta) = \hat{B}I(z,\theta), \tag{7}$$

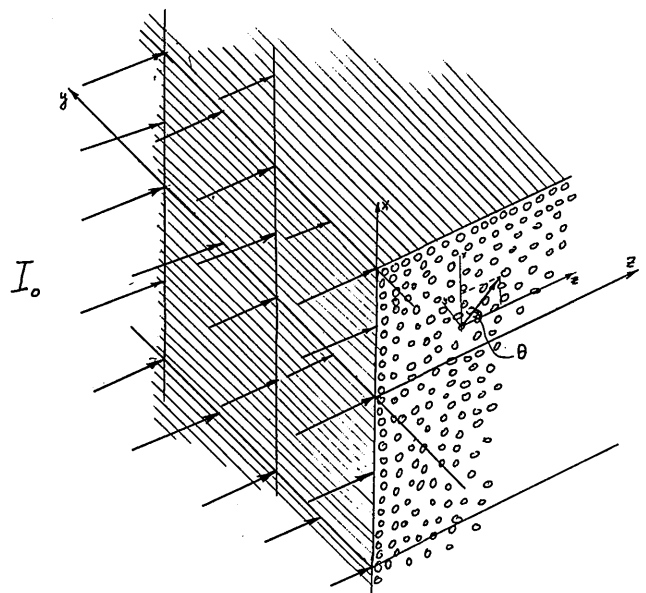
Fig. 1. Schematic showing main physical quantities, variables, and geometry of the problem.

where $\hat{B}I(z,\theta)$ is the linearized Boltzmann collision integral,

$$\hat{B}I(z,\theta) = \sigma \int_{-\pi}^{\pi} \mathrm{d}\theta'\chi(\gamma)[I(z,\theta') - I(z,\theta)], \qquad (8)$$

and θ is the angle between the direction in which the radiation propagates and the yOz plane. In the 2D case, $\gamma = \theta' - \theta$ is the angle of single scattering from θ' to θ.

Below we consider only the simplest case, from the point of view of analysis, of an initial flux normally incident upon the boundary of the medium. The boundary condition for Eqs. (7) and (8) will then be the following:

$$I(z = 0, \theta) = I_0\delta(\theta). \qquad (9)$$

Strictly speaking, condition (9) holds only for incident radiation, i.e., in the range of angles $-\pi/2 < \theta < \pi/2$. However, in the case of strong absorption, when multiple scattering involves mainly small angles, the backward-scattered radiation can be neglected, and condition (9) can be extended to all angles: $-\pi < \theta < \pi$. Taking into account that practically all the radiation is concentrated within the region of small angles, one may formally consider all angular variables to vary within infinite limits: $-\infty < \theta, \gamma < \infty$.

Let us single out in the intensity being sought the exponential attenuation of the beam, which takes into account the absorption of light in the medium,

$$I(z,\theta) = I_0 \exp(-\kappa z)\tilde{I}(z,\theta). \qquad (10)$$

Then, substituting Eq. (10) into Eqs. (7) and (8) and expanding, in small angles θ up to the first nonnegligible terms, all the coefficients that occur in the equation obtained for $\tilde{I}(z,\theta)$, we obtain the transfer equation in the small-angle approximation with account taken of the influence of absorption on the photon angular distribution:

$$\frac{\partial \tilde{I}(z,\theta)}{\partial z} + \frac{\kappa}{2}\theta^2\tilde{I}(z,\theta) = \hat{B}\tilde{I}(z,\theta). \qquad (11)$$

Now the elastic collision integral assumes, within the small-angle approximation, the following form:

$$\hat{B}\tilde{I}(z,\theta) = \sigma \int_{-\infty}^{\infty} \mathrm{d}\theta'\chi(\gamma)[\tilde{I}(z,\theta') - \tilde{I}(z,\theta)]. \qquad (12)$$

Equation (11) differs from the small-angle-approximation transfer equation[2,5-11] by the term $\kappa(\theta^2/2)\tilde{I}$ that takes into account the fluctuations of photon paths. This term provides the effect of absorption on multiple scattering and is of particular importance at large depths.

3.   EXACT SOLUTION

For solution of the set of Eqs. (11), (12), and (9), we take advantage of the Fourier transformation over the angle θ:

$$\tilde{I}(z,\theta) = \frac{1}{\pi}\int_0^{\infty} \cos \omega\theta\tilde{I}(z,\omega)\mathrm{d}\omega. \qquad (13)$$

We shall search for the Fourier transform of the intensity $\tilde{I}(z,\omega)$ by the method of separation of variables. In this case $\tilde{I}(z,\omega)$ can be represented in the form

$$\tilde{I}(z,\omega) = \sum_{m=0}^{M} c_m\Phi_m(\omega)\exp(-k_mz)$$
$$+ \int_0^{\infty} c(\mu)\Phi_\mu(\omega)\exp(-k_\mu z)\mathrm{d}\mu. \qquad (14)$$

The functions $\Phi_{m,\mu}$ are eigenfunctions of the equation

$$\frac{\mathrm{d}^2\Phi_{m,\mu}(q)}{\mathrm{d}q^2} = \{\eta[1 - \chi(q)] - \varepsilon_{m,\mu}\}\Phi_{m,\mu}(q), \qquad (15)$$

and

$$\varepsilon_{m,\mu} = \eta\frac{k_{m,\mu}}{\sigma} \qquad (16)$$

are its eigenvalues. Here, instead of ω, the more convenient variable

$$q = \omega\gamma_{ef} \qquad (17)$$

is introduced, as well as the parameter

$$\eta = \frac{2\sigma}{\kappa\gamma_{ef}^2}. \qquad (18)$$

The first term in Eq. (14) determines the contribution to $\tilde{I}(z,\omega)$ of the discrete spectrum of Eq. (15); it involves M eigenvalues. The second term determines the contribution of the continuous spectrum.

Boundary condition (9) results in the equality

$$\sum_{m=0}^{M} c_m\Phi_m(q) + \int_0^{\infty} c(\mu)\Phi_\mu(q)\mathrm{d}\mu \equiv 1. \qquad (19)$$

Taking advantage of the eigenfunctions $\Phi_{m,\mu}$ that exhibit the property of orthogonality, one may readily determine from Eq. (19) the coefficients of the expansion, c_m and $c(\mu)$. Thus for the coefficients c_m we obtain

$$c_m = \frac{\int_{-\infty}^{\infty} \Phi_m(q)\mathrm{d}q}{\int_{-\infty}^{\infty} \Phi_m^2(q)\mathrm{d}q}. \qquad (20)$$

In the determination of the values of $c(\mu)$ the normalizing integral diverges, so in this case it is convenient to make use of the asymptotic representation $\Phi_\mu(q)$, which permits the use of the respective δ functions.[26]

The Fourier transform of the Heney–Greenstein phase functions [relation (1)] assumes the following form in the notation of Eq. (17):

$$\chi^{HG}(q) = \exp(-|q|). \qquad (21)$$

In accordance with the parity of $\chi^{HG}(q)$, the solution of Eq. (15) for each eigenvalue $\varepsilon_{m,\mu}$ exhibits a certain parity: it may be either even or odd. However, in the case of a beam normally incident upon the surface of a medium, the intensity of the incident radiation is an even function of the angle θ, and its Fourier transform is an even function of q. Therefore the odd part of the spectrum does not contribute to expansion (14), as can be seen from Eq. (20).

The parity of $\Phi_{m,\mu}(q)$, together with the requirement that the derivative of this function exist with respect to q,
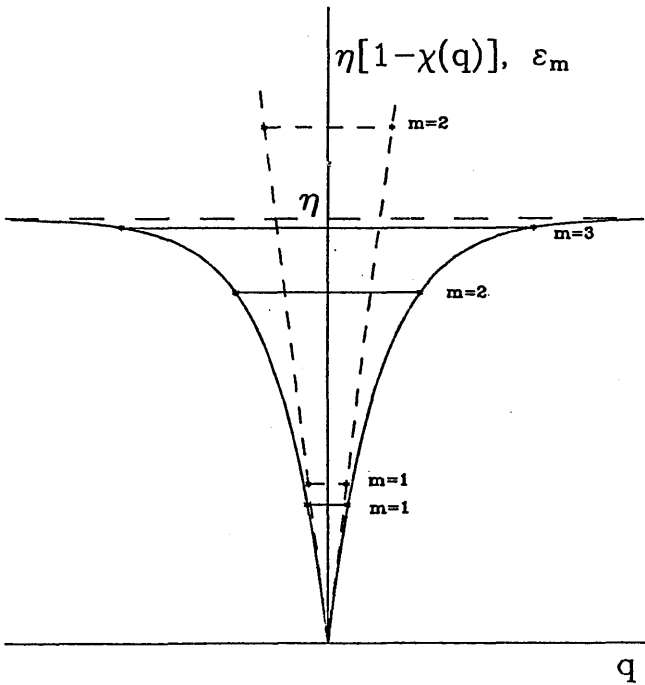
Fig. 2. Quantum-mechanical potentials $U(q) = \eta[1 - \chi(q)]$ and energy levels ε_m corresponding to the exact boundary problem (solid curves) and the boundary problem in the QDA (dashed curves).

leads, for $q = 0$, to the condition

$$\left(\frac{d\Phi_{m,\mu}}{dq}\right)_{q=0} = 0. \tag{22}$$

The system consisting of Eqs. (15) and (22) must be supplemented also by the requirement that a restriction be imposed on $\Phi_{m,\mu}(q)$.

The obtained boundary problem is equivalent to the well-known quantum-mechanical problem of the one-dimensional motion of a particle of energy $\varepsilon_{m,\mu}$ in a well with the potential $U(x) = U_0[1 - \exp(-|x|)]$ (the solid curves of Fig. 2), the solution of which is presented, for example, in Ref. 27. In agreement with Ref. 27, the expression for the functions $\Phi_{m,\mu}$ have the form

$$\Phi_m(q) = J_{2\sqrt{\eta - \varepsilon_m}}\left[2\sqrt{\eta}\exp\left(-\frac{|q|}{2}\right)\right], \tag{23}$$

$$\Phi_\mu(q) = J_{i\mu}\left[2\sqrt{\eta}\exp\left(-\frac{|q|}{2}\right)\right]$$
$$+ a_\mu J_{-i\mu}\left[2\sqrt{\eta}\exp\left(-\frac{|q|}{2}\right)\right], \tag{24}$$

where $J_\nu(x)$ is the Bessel function and $\mu = 2\sqrt{\varepsilon_\mu - \eta}$ (note that $\varepsilon_\mu > \eta$).

Now, calculating c_m and $c(\mu)$, we obtain the final expression for the Fourier transform of the intensity $\tilde{I}(z, q)$:

$$\tilde{I}(z, q) = \tilde{I}_{\text{discr}}(z, q) + \tilde{I}_{\text{cont}}(z, q), \tag{25}$$

where $\tilde{I}_{\text{discr}}$ and $\tilde{I}_{\text{cont}}$ represent the contributions of the discrete and the continuous spectra, respectively.

The function $\tilde{I}_{\text{discr}}(z, q)$ is defined by the following formula:

$$\tilde{I}_{\text{discr}}(z, q) = \sum_{m=0}^{M} c_m J_{\xi m}\left[2\sqrt{\eta}\exp\left(-\frac{|q|}{2}\right)\right]\exp(-k_m z), \tag{26}$$

where $k_m(\eta)$ is expressed through ε_m from Eq. (16):

$$c_m = 2\frac{\Gamma(\xi_m + 1)}{\eta^{\xi_m/2}}$$
$$\times \frac{{}_1F_2(\xi_m/2; \xi_m/2 + 1, \xi_m + 1; -\eta)}{{}_2F_3(\xi_m + \frac{1}{2}, \xi_m; \xi_m + 1, \xi_m + 1, 2\xi_m + 1; -4\eta)}, \tag{27}$$

where ${}_1F_2$ and ${}_2F_3$ are hypergeometric functions and the quantity ξ_m is the index of the Bessel function in Eq. (23):

$$\xi_m = 2\sqrt{\eta - \varepsilon_m}. \tag{28}$$

We recall that the hypergeometric functions ${}_1F_2$ and ${}_2F_3$ are particular cases of the generalized hypergeometric function ${}_pF_q$, defined by the series[28]

$$_pF_q[(a_p); (b_q); x] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{x^k}{k!},$$

where $b_j \neq 0, -1, -2, \ldots; j = 1, \ldots, q$. Here $(a)_k = a(a + 1)\ldots(a + k - 1) = \Gamma(a + k)/\Gamma(a), (a)_0 = 1$. When $p < q$, this series converges for all x, whereas if $p = q + 1$, the series converges under the condition that $|x| < 1$. In the particular case of $p = 2$, $q = 1$ we have the well-known hypergeometric Gaussian function ${}_2F_1$, and for $p = q = 1$ we have Kummer's degenerate hypergeometric function ${}_1F_1$.

Eigenvalues of the discrete spectrum $\varepsilon_m(\eta)$ [or $\xi_m(\eta)$] are determined from condition (22), which, if we take into account Eq. (23), assumes the form:

$$J_{\xi_m}'(2\sqrt{\eta}) = 0, \tag{29}$$

where $J_\nu'(x)$ is the derivative of the Bessel function with respect to its argument.

The dependences of the first several $(m = 0, 1, 2)$ eigenvalues ξ_m on η are presented in Fig. 3. Note that the number of roots M of Eq. (29) (the number of levels in
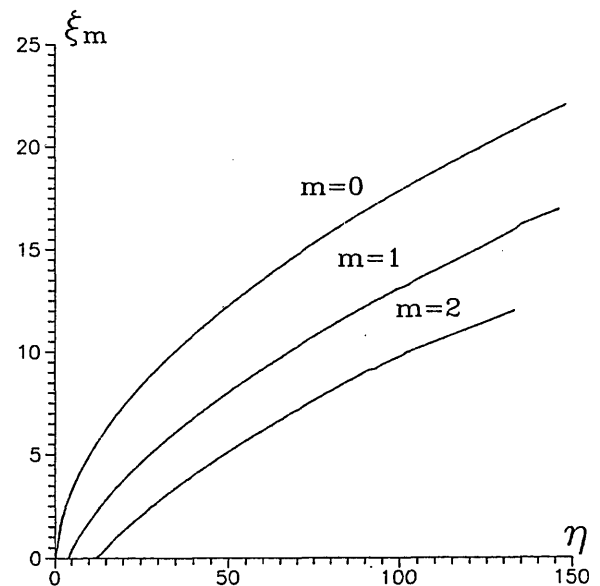

Fig. 3. Dependences of the first several $(m = 0, 1, 2)$ eigenvalues ξ_m on η, determined from condition (29).

the discrete spectrum) increases with the parameter η [Eq. (18)], which is the principal parameter of the problem: $M = M(\eta)$; $\varepsilon_m = \varepsilon_m(\eta)$.

If we make use of the asymptotic representation of the Bessel function in the case when its argument exceeds the index significantly, it is possible to obtain the following estimate for $M(\eta)$:

$$M = \text{INT}\left(\frac{2}{\pi}\sqrt{\eta} + \frac{3}{4}\right), \tag{30}$$

where $\text{INT}(x)$ is the integer part of x. Comparison with exact calculations reveals that formula (30) turns out to be very precise even in the region of relatively small η, when $M = 2, 3$. One (the first) root of Eq. (29) exists at all η, which reflects the general property of the spectrum of a one-dimensional boundary problem.[26]

Regarding the continuous spectrum, the quantities a_μ occurring in Eq. (24) are determined from condition (22):

$$a_\mu = -\frac{J_{i\mu}'(2\sqrt{\eta})}{J_{-i\mu}'(2\sqrt{\eta})}. \tag{31}$$

Now, substituting Eq. (31) into Eq. (24) and taking account of Eq. (19), we obtain, on performing simple manipulations, the following expression for $\tilde{I}_{\text{cont}}(z,q)$:

$$\tilde{I}_{\text{cont}}(z,q) = \frac{\exp(-\sigma z)}{2}\int_{-\infty}^{\infty}\frac{\mu d\mu}{\sinh\pi\mu}\frac{Q_\mu}{a_\mu}$$

$$\times J_{i\mu}\left[2\sqrt{\eta}\exp\left(-\frac{|q|}{2}\right)\right]\exp\left(-\frac{\sigma}{\eta}\mu^2 z\right). \tag{32}$$

Here

$$Q_\mu = Q_\mu^+ + a_\mu Q_\mu^-, \tag{33}$$

where

$$Q_\mu^\pm = -\frac{2i}{\mu}\frac{\eta^{\pm i\mu/2}}{\Gamma(1\pm i\mu)}\,{}_1F_2\left(\pm\frac{i\mu}{2}; 1\pm\frac{i\mu}{2}, 1\pm i\mu; -\eta\right). \tag{34}$$

Note that, although Eq. (31) contains complex quantities, $\tilde{I}_{\text{cont}}(z,q)$ itself is a real function, since $a_\mu^* = a_{-\mu}$; $Q_\mu^* = Q_{-\mu}$, $J_{i\mu}^* = J_{-i\mu}$. Expanding the Bessel function in Eq. (32) and performing Fourier transformations in Eqs. (26) and (31), we obtain the final expression for the light intensity in the medium at a depth z in the following form:

$$I(z,\theta) = \frac{I_0}{\pi\gamma_{ef}}\exp[-(\kappa + \sigma)z]$$

$$\times \left\{\sum_{m=0}^{M(\eta)}\frac{\eta^{\xi_m/2}}{\Gamma(\xi_m + 1)}c_m\sum_{n=0}^{\infty}\frac{(-\eta)^n}{(\xi_m + 1)_n n!}\right.$$

$$\times\left[\frac{\xi_m/2 + n}{(\xi_m/2 + n)^2 + (\theta/\gamma_{ef})^2}\right]\exp\left[\frac{\sigma}{\eta}(\eta - \varepsilon_m)z\right]$$

$$\left. + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-\eta)^n}{n!}A_n\left(z, \psi = \frac{\theta}{\gamma_{ef}}\right)\right\}, \tag{35}$$

where

$$A_n(z,\psi) = \int_{-\infty}^{\infty}\frac{\mu d\mu}{\sinh\pi\mu}\frac{Q_\mu}{a_\mu}\frac{\eta^{i\mu/2}}{\Gamma(n + 1 + i\mu)}$$

$$\times\left[\frac{n + i\mu/2}{(n + i\mu/2)^2 + \psi^2}\right]\exp\left(-\frac{\sigma}{\eta}\mu^2 z\right) \tag{36}$$

(A_n is a real quantity).

Analytical and numerical investigations of the angular functions occurring in the discrete spectrum reveal that, as $\theta \to \infty$, the angular functions decrease as θ^{-4}, i.e., with a power index exceeding 1, with which the phase function decreases, by 2.

The solution presented is based only on the conditions of applicability of the small-angle approximation [Eq. (6)]. Formulas (35) and (36) do not imply multiple scattering. These formulas are applicable, also, at lesser depths, where the main role is assumed by radiation that is not undergoing any scattering or is undergoing only single scattering.

4. SHARPLY ANISOTROPIC SCATTERING IN THICK LAYERS: THE QUASI-DIFFUSION APPROXIMATION

We consider the quasi-diffusion approximation (QDA) to be a natural generalization of the diffusion approximation in the angular variable (the Fokker–Planck approximation) to the case of scattering phase functions decreasing in the 2D case more slowly than γ^{-3}.

The quasi-diffusion approximation essentially consists in the Fourier transform of the phase function in Eq. (15) being replaced by the first term of its expansion in small q. This corresponds to the phase function of the form of Eq. (2), occurring in collision integral (12), being replaced by its asymptote $\sim\gamma^{-(2\nu+1)}$.

The condition to be satisfied for application of the QDA is the same as for the diffusion approximation: the possibility of expansion in small q is determined by the smallness of the effective single-scattering angle γ_{ef} as compared with the characteristic multiple-scattering angle. This condition implies that the scattering is essentially multiple and is satisfied in thick layers ($z \gg l$, where $l = \sigma^{-1}$ is the scattering length).

The quantum-mechanical analogy of one-dimensional motion in the potential $U(x) = U_0|x|$ (dashed curve of Fig. 2) corresponds to the transport problem within the QDA. In the case of phase function (2), $U(x) = U_0|x|^{2\nu}$. The solution for the linear potential $U(x)$ is well known and is given in Ref. 27.

Unlike in the exact case, no continuous spectrum is present in the model based on the QDA, while the discrete spectrum exhibits an infinite number of levels. Therefore it is evident, beforehand, that the QDA is justified only in the case of a sufficiently large number of eigenvalues of the discrete spectrum, the contribution of which, in practice, totally determines the radiation intensity in the medium. In the QDA the contribution of levels belonging to the discrete spectrum, with $\varepsilon_m \sim \eta$, and of the continuous spectrum are considered negligible.

In accordance with Eq. (30), the above assumptions result in the condition

$$\eta \gg 1, \tag{37a}$$

which in the usual notation assumes the form

$$l_a l_{tr}{}^2 \gg l^3. \tag{37b}$$

Taking into account the equality $l_{tr} = l/(1 - \langle\cos\gamma\rangle)$, we obtain the following restriction on the degree of elongation of the phase function:

$$\left(\frac{\sigma}{\kappa}\right)^{1/2} = \left(\frac{l_a}{l}\right)^{1/2} \gg (1 - \langle\cos\gamma\rangle). \tag{37c}$$

For characteristic values of $\sigma/\kappa = l_a/l_{tr} \geq 1 \div 3$, $\gamma_{ef} \sim 0.1$, we obtain from Eq. (18) that $\eta \geq 200 \div 600$ and that condition (37) is satisfied with a large safety margin.

Now, bearing in mind the structure of the spectrum of the problem within the QDA, we proceed to look for the Fourier transform of intensity (13) in the form

$$\tilde{I}^{\text{QDA}}(z, q) = \sum_{m=0}^{\infty} c_m \Phi_m(q) \exp(-k_m z) \tag{38}$$

instead of that of Eq. (14). Here the eigenfunctions $\Phi_m(q)$ are determined by the equation obtained from Eq. (15) by substitution, for the exact expression of Eq. (21) for $\chi^{HG}(q)$, of the approximate expression $\chi^{HG}(q) \approx |q|$:

$$\frac{d^2 \Phi_m(q)}{dq^2} = (\eta|q| - \varepsilon_m)\Phi_m(q). \tag{39}$$

As shown above, the parameter ε_m is determined by formula (16) and plays the part of energy in the appropriate quantum-mechanical problem. Condition (22) retains its effect in the QDA. Boundary condition (9) leads to an expression that differs from Eq. (19) only in that no contribution of the continuous spectrum is present and by the substitution of ∞ for M.

The expression for the eigenfunctions of problem (39) has the form[27]

$$\Phi_m(q) = Ai(\eta^{1/3}|q| - \beta_m), \tag{40}$$

where $Ai(x)$ is the Airy function and β_m is determined from condition (22), which for the functions in Eq. (40) assumes the form

$$Ai'(-\beta_m) = 0. \tag{41}$$

In accordance with Eq. (41), the $(-\beta_m)$ coincide with the positions of the extrema of the Airy function on the negative semiaxis. We can solve Eq. (41) approximately, making use of the asymptotic representation of the Airy function. In this case, for $m \geq 1$,

$$\beta_m \approx \left[\frac{3}{2}\pi\left(m - \frac{3}{4}\right)\right]^{2/3}, \tag{42}$$

and $\beta_0 \approx 1.019$.

Further, determining the coefficients c_m,

$$c_m = \frac{1}{\beta_m Ai^2(-\beta_m)} \int_{-\beta_m}^{\infty} Ai(x)dx, \tag{43}$$

and performing Fourier transformation, we find the following expression for the radiation intensity $I(z, \theta)$ within the QDA:

$$I^{\text{QDA}}(z, \theta) = I_0 \frac{\exp(-\kappa z)}{\gamma_{ef}} \sum_{m=0}^{\infty} c_m P_{\beta_m}\left(\frac{\theta}{\gamma_{ef}\eta^{1/3}}\right)\exp(-k_m z), \tag{44}$$

where

$$P_\beta(x) = \frac{1}{\pi\eta^{1/3}} \int_0^{\infty} Ai(y - \beta)\cos(xy)dy, \tag{45}$$

$$k_m = \beta_m \eta^{1/3} \sigma. \tag{46}$$

Numerical studies show that, as in the exact solution (Section 3), the angular functions in expansion (44) decrease as θ^{-4}, as θ increases.

The results of Eqs. (44)–(46) are obtained by solution of the transformed transfer equation, the equation involving the elastic collision integral in the QDA, and therefore the validity of Eqs. (44)–(46) is based on the feasibility of using, in Eq. (12), the asymptote of the phase function for relatively large scattering angles [or, which is the same, of expanding, in Eq. (15), the function $\chi(q)$ in a series up to the term linear in q]. At the same time, having at our disposal the exact solution [Eqs. (35) and (36)] of the initial transfer equation [Eqs. (11) and (12)], we can check how distribution (40) is obtained from Eq. (23). It is also possible, in the course of such transition, to clarify the range of applicability of the solution within QDA (37).

The following assertions are valid in conditions of multiple scattering. First, the main contribution to expression (25), in the case of $z \gg l$, is due to terms corresponding to eigenvalues of the discrete spectrum, namely, to eigenvalues $\varepsilon_m \ll \eta(\eta \gg 1)$. Second, the index ξ_m of the Bessel function, through which the eigenfunctions $\Phi_m(q)$ are expressed [Eq. (23)], turns out to be large and of the order of $\sqrt{\eta}$ when $\eta \gg 1$. Third, the angle θ of multiple photon scattering exceeds the effective angle of single deviation $\gamma_{ef}(\theta \gg \gamma_{ef})$, and therefore the value of q in Eq. (15) can be considered small ($q \ll 1$). The above assertions allow one, by utilizing the asymptotic Nicholson formula for the Bessel function,[29] to expand the argument of Eq. (23) in $|q|$ and the condition $\varepsilon \ll \eta$ to obtain for $\Phi_m(q)$ the following approximate representation:

$$\Phi_m(q) = J_{\xi_m}\left[2\sqrt{\eta}\exp\left(-\frac{|q|}{2}\right)\right] \approx \frac{1}{\eta^{1/6}\exp(-|q|/6)}$$

$$\times Ai\left[\frac{\xi_m - 2\sqrt{\eta}\exp(-|q|/2)}{\eta^{1/6}\exp(-|q|/6)}\right]$$

$$\approx \frac{1}{\eta^{1/6}} Ai(\eta^{1/3}|q| - \beta_m), \tag{47}$$

where

$$\beta_m = \frac{2\sqrt{\eta} - 2\sqrt{\eta - \xi_m}}{\eta^{1/6}} \approx \frac{\varepsilon_m}{\eta^{2/3}}. \tag{48}$$

Expression (47) coincides, up to an insignificant common factor, with Eq. (40), while substitution of Eq. (16) into expression (48) leads to an expression identical to Eq. (46).

It is important to stress that the significance of approximate equalities (47) and (48) consists not only in the justification of transition from the exact solution [Eq. (23)] to the approximate one [expression (47)] but also in the possibility of determining corrections to the approximate solution with the aid of approximate equalities (47) and (48).

5. DISCUSSION OF THE RESULTS: THE DEPTH MODE

At a large depth z only a sole term corresponding to the minimal decay coefficient k_0 survives in expansions (14) and (38). Thus factorization of the solution takes place: it becomes possible to represent the intensity $I(z, \theta)$ in the form of the product of two functions, one of which depends only on the depth z, while the other depends only on the scattering angle θ. In experiments the onset of an asymptotic depth mode of light propagation manifests itself in the angular spectrum's no longer varying with the depth.

Note that simplification of the form of the solution of the transfer equation in the depth mode significantly extends the possibilities of application of various approximate and numerical methods.[5,13-16]

The conditions for the onset of the depth mode can be obtained from comparison of the exponents occurring in the first and second terms of expansions (14) and (37):

$$z \gg \frac{1}{k_1 - k_0}.\qquad(49)$$

Within the framework of the QDA, formula (46) makes inequality (49) assume the form

$$\sigma z \gg \eta^{1/3},\qquad(50)$$

or, in dimensional units,

$$z \gg \sqrt[3]{l_a l_{tr}^2}.\qquad(51)$$

It is interesting that numerical investigation of the exact solution reveals condition (50) to hold also when the QDA is not applicable.

The depth attenuation coefficient k_0 is determined from Eq. (16), where ε_0 corresponds to the smallest root of Eq. (29). Expanding the first root of the derivative of the Bessel function with respect to its large index,[29] which is valid for $\eta \gg 1$, one can obtain the following asymptotic expression for k_0:

$$k_0 = \frac{\sigma}{\eta}(1.019\eta^{2/3} - 0.375\eta^{1/3} + 0.033).\qquad(52)$$

Numerical investigations show this formula to yield a result coinciding with the exact one not only for large η but for practically all values from the domain of this parameter (up to $\eta \sim 0.1$).

The first term of expansion (52) coincides with the QDA result [see formula (46)]:

$$k_0^{QDA} = 1.019\sigma\eta^{-1/3}.\qquad(53)$$

A similar statement holds also for the remaining k_m for $m \geq 1$.

Comparing formulas (52) and (53), one can easily obtain the applicability condition of the QDA for describing the radiation intensity in the depth mode:

$$z \ll \frac{1}{k_0^{QDA} - k_0},\qquad(54)$$

or, taking into account Eqs. (52) and (53) and expression (50),

$$\eta^{1/3} \ll \sigma z \ll \eta^{2/3}.\qquad(55)$$

In accordance with the last inequality, the condition $\eta \gg 1$ [expression (37a)] is sufficient for a region of depths to exist in which the QDA is applicable.

In Fig. 4 plots are given of the dependences k_0/σ (solid curve) and k_0^{QDA}/σ (dashed curve). In spite of the fact that, when $\eta = 0$, $k_0 = \sigma$ and $k_0^{QDA} = \infty$, one can see from the plots that the difference between them, already for $\eta \geq 10$, does not exceed 10%. The QDA yields a value for the depth attenuation coefficient that is somewhat elevated compared with that of the exact coefficient.

The angular distribution of the radiation intensity in the depth mode can be readily obtained from the exact solution [Eq. (35)] or, within the framework of the QDA, from Eq. (44). Since factorization of the angular spectrum of the radiation occurs in the depth mode, it is convenient, together with the intensity $I_\infty(z, \theta)$, to introduce the angular function $\Phi_\infty(\theta) = (1/\gamma_{ef})f(\theta/\gamma_{ef})$, normalized by the condition

$$\int_{-\infty}^{\infty} \Phi_\infty(\theta)d\theta = \int_{-\infty}^{\infty} f(\psi)d\psi = 1.\qquad(56)$$

The quantity $\Phi_\infty(\theta)$ may be interpreted as the probability density of photon scattering through angles between θ and $\theta + d\theta$.

Then $I_\infty(z, \theta)$ may be represented in the form

$$I_\infty(z, \theta) = E_\infty(z)\Phi_\infty(\theta),\qquad(57)$$

where

$$E_\infty(z) = I_0 g_\infty \exp[-(\kappa + k_0)z]\qquad(58)$$

is the total radiation flux in the depth mode ($\sigma z \gg \eta^{1/3}$).

The explicit expressions for $f(\psi)$ and g_∞ are given below. In accordance with Eqs. (35) and (27), g_∞ and $f(\psi)$, corre-
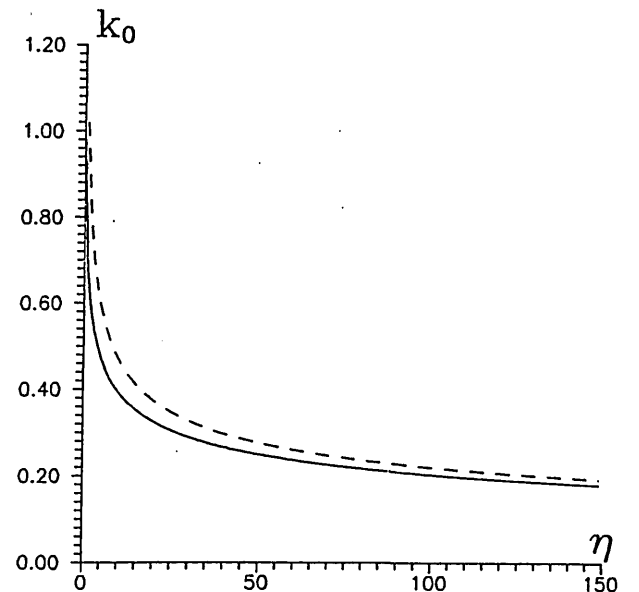

Fig. 4. Dependences of the ratio between the depth attenuation coefficient k_0 and the scattering coefficient σ on η in the exact problem (solid curve) and in the problem in the QDA (dashed curve).
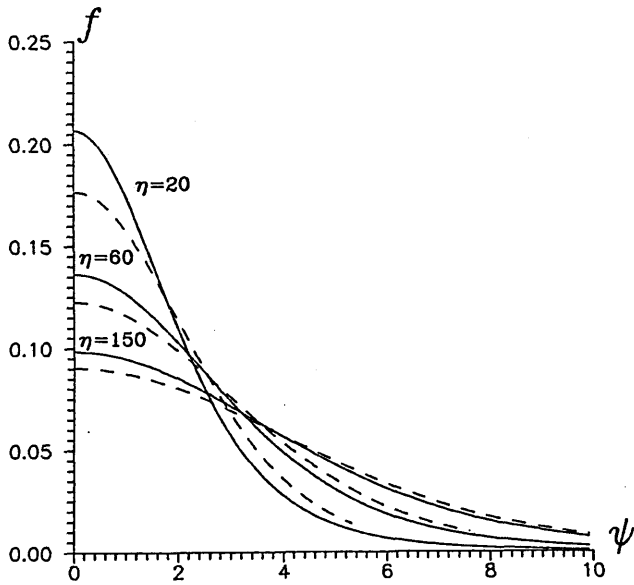
Fig. 5. Plots of the depth angular distribution functions $f(\psi)$ and $f^{\text{QDA}}(\psi)$ for $\eta = 20$, 60, and 150.

sponding to the exact solution, have the forms:

$$g_\infty = 2\frac{{}_1F_2(\xi_0/2;\xi_0/2+1,\xi_0+1;-\eta)\,{}_0F_1(\xi_0+1;-\eta)}{{}_2F_3(\xi_0+\frac{1}{2},\xi_0;\xi_0+1,\xi_0+1,2\xi_0+1;-4\eta)},$$

(59)

$$f(\psi) = \frac{1}{\pi\,{}_0F_1(\xi_0+1;-\eta)}\sum_{n=0}^{\infty}\frac{(-\eta)^n}{(\xi_0+1)_n n!}$$
$$\times\left[\frac{\xi_0/2+n}{(\xi_0/2+n)^2+\psi^2}\right],$$

(60)

where ξ_0 is determined by Eq. (28).

From Eq. (44) we have the following for the same quantities within the QDA:

$$g_\infty^{\text{QDA}} = c_0 Ai(-\beta_0) \approx 1.47,$$

(61)

$$f^{\text{QDA}}(\psi) = \frac{1}{Ai(-\beta_0)}P_{\beta_0}(\psi\eta^{-1/3}),$$

(62)

where $P_\beta(x)$ is determined by formula (45).

From a comparison of Eqs. (60) and (62) it follows that, when the QDA is applied, a qualitative property of the exact solution is lost: the dependence of g_∞ on η. Truly, this dependence is very weak: g_∞ decreases slowly as η increases, and in the region of large η (starting from $\eta \geq 10$) of interest, it lies in the following interval: $g_\infty \approx 1.53 \div 1.47$.

We noted above that the angular functions pertaining to the discrete spectrum decrease as θ^{-4}, as θ increases. That statement is relevant also to the angular spectrum in the depth mode. It is easily seen, in particular, if we rewrite Eq. (60), taking into account Eq. (29) in the form

$$f(\theta) = \frac{1}{\pi\,{}_0F_1(\xi_0+1;-\eta)}\sum_{n=0}^{\infty}\frac{(-\eta)^n}{(\xi_0+1)_n n!}$$
$$\times\frac{(\xi_0/2+n)^3}{[(\xi_0/2+n)^2+\psi^2]\psi^2}.$$

(63)

The aymptote of this function is

$$f(\psi\to\infty) = \frac{b}{\psi 4},$$

(64)

where

$$b = \frac{\eta}{3\pi}.$$

(65)

The dependences $f(\psi)$ and $f^{\text{QDA}}(\psi)$ are presented in Fig. 5. It can be clearly seen that, when $\psi = 0$, the curves $f^{\text{QDA}}(\psi)$ differ from the exact ones by 10–15%. These differences decrease as η increases.

From Fig. 5 one can also note the increase, with the growth of η, of the width of the angular spectrum. Such behavior can be revealed with the aid of analytical calculations, as well.

It is known that the dispersion of the angular distribution is calculated with the aid of the formula

$$\langle\theta^2\rangle_z = \int_{-\infty}^{\infty}\theta^2 I(z,\theta)\mathrm{d}\theta\Big/\int_{-\infty}^{\infty}I(z,\theta)\mathrm{d}\theta,$$

(66)

which one may transform, taking into account Eqs. (56) and (57), into the form

$$\langle\theta^2\rangle_\infty = \gamma_{ef}^2\int_0^{\infty}\psi^2 f(\psi)\mathrm{d}\psi.$$

(67)

Although the second moment of phase function (1) does not exist, the mean square of multiple-scattering angle (67) is a confined value. This is due to the effect of light absorption in the medium. The process of absorption suppresses multiple scattering through relatively large angles and provides more rapid decrease of $f(\psi)$ [expression (64)] as compared with the phase function [expression (1)]. Substitution of Eq. (63) into Eq. (67) with account taken of Eq. (16) yields the relation between the depth attenuation coefficient k_0 and the dispersion $\langle\theta^2\rangle_\infty$:

$$k_0 = \frac{\kappa}{2}\langle\theta^2\rangle_\infty,$$

(68)

and we may obtain the expansion of $\langle\theta^2\rangle_\infty$ in powers of η:

$$\langle\theta^2\rangle_\infty = \gamma_{ef}^2(1.019\eta^{2/3} - 0.375\eta^{1/3} + 0.033).$$

(69)

Within the framework of the QDA the value of $\langle\theta^2\rangle_\infty$ is determined by the first term of Eq. (69):

$$\langle\theta^2\rangle_\infty^{\text{QDA}} = 1.019\eta^{2/3}\gamma_{ef}^2.$$

(70)

As η increases, the applicability condition $\langle\theta^2\rangle_\infty \ll 1$ [expression (6)] for the small-angle approximation is violated. This circumstance imposes an upper limit on the value of the parameter η:

$$\eta \ll \gamma_{ef}^{-3},$$

(71)

or

$$l_a \ll l_{tr}.$$

(72)

The latter condition coincides with the initial requirement of strong absorption [expression (5)] underlying the small-angle mode of light propagation at all depths.

In closing, it may be noted that not only does the exactly solvable problem considered above enable one to under-

stand the laws of light propagation in an absorbing medium, but it is also applicable as a good test for various approximate methods of solving the transfer equation.

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REFERENCES

1. A. Isimaru, *Wave Propagation and Scattering in Random Media* (Academic, New York, 1978).
2. K. M. Case and P. F. Zweifel, *Linear Transport Theory* (Addison-Wesley, Reading, Mass., 1967).
3. N. P. Kalashnikov, V. S. Remizovich, and M. I. Ryazanov, *Collisions of Fast Charged Particles in Solids* (Gordon & Breach, New York, 1985).
4. V. S. Remizovich, D. B. Rogozkin, and M. I. Ryazanov, *Range Fluctuations of Charged Particles* (Energoatomizdat, Moscow, 1988).
5. E. P. Zege, A. P. Ivanov, and I. L. Katsev, *Image Transfer in the Scattering Medium* (Nauka i Tekhnika, Minsk, USSR, 1985).
6. S. Goudsmit and J. Sounderson, "Multiple scattering of electrons," Phys. Rev. **57**, 24–33 (1940).
7. G. Molière, "Theorie der Streuung shneller gelandener Teilchen I. Einzelstreuung am adgeschirmten Goulomb-Feld," Z. Naturforsh. **2a**, 133–145 (1947).
8. W. T. Scott, "The theory of small-angle multiple scattering of fast charged particles," Rev. Mod. Phys. **35**, 231–313 (1963).
9. L. S. Doliin, "On light beam scattering in a slab of a turbid medium," Izv. Vyssh. Uchebn. Zaved. Radiofiz. **7**, 380–382 (1964).
10. L. S. Dolin, "On propagation of a narrow light beam in a medium with strongly anisotropic scattering," Izv. Vyssh. Uchebn. Zaved. Radiofiz. **9**, 61–71 (1966).
11. R. L. Fante, "Propagation of electromagnetic waves through turbulent plasma using transport theory," IEEE Trans. Antennas Propag. **21**, 750–755 (1973).
12. L. S. Dolin, "Automodel approximation in the theory of multiple sharply anisotropic scattering of light," Dokl. Akad. Nauk SSSR **260**, 1344–1347 (1981).
13. V. S. Remizovich, D. B. Rogozkin, and M. I. Ryazanov, "Propagation of a light signal in a substance with large-scale random inhomogeneities with photon path-length fluctuations due to multiple scattering taken into account," Izv. Akad. Sci. USSR, Atmos. Ocean. Phys. **18**, 480–485 (1982).
14. V. S. Remizovich, D. B. Rogozkin, and M. I. Ryazanov, "Propagation of a narrow modulated light beam in a scattering medium with fluctuations of the photon pathlength in multiple scattering," Radiophys. Quantum Electron. **25**, 639–645 (1982).
15. V. S. Remizovich, D. B Rogozkin, and M. I. Ryazanov, "Propagation of a light-flash signal in a turbid medium," Izv. Acad. Sci. USSR, Atmos. Ocean. Phys. **19**, 796–801 (1983).
16. M. D. Alexandrov and V. S. Remizovich, "Calculation of light field characteristics in the depth mode of real turbid media," Opt. Atmos. **4**, 1017–1024 (1991).
17. S. V. Maleyev and B. P. Toperverg, "Small-angle multiple scattering by statical inhomogeneities," Zh. Eksp. Teor. Fiz. **78**, 315–330 (1980).
18. V. S. Remizovich, "Theoretical discription of elastic reflection of particles (photons) incident at grazing angles without the use of the diffusion approximation," Sov. Phys. JETP **60**, 290–297 (1984).
19. V. S. Remizovich, D. B. Rogozkin, and Sh. A. Shekhmametyev, "Calculation of characteristics of scattered light within the self-similar diffusion approximation," Izv. Akad. Sci. USSR, Atmos. Ocean. Phys. **27**, 196–202 (1991).
20. R. Berkovits and M. Kaveh, "Weak localisation of photons in a disordered two-dimensional medium," J. Phys. C **20**, L181–L185 (1987).
21. I. Freund, M. Rosenbluh, R. Berkovits, and M. Kaveh, "Coherent backscattering of light in a quasi-two-dimensional system," Phys. Rev. Lett. **61**, 1214–1217 (1988).
22. E. E. Gorodnichev, S. L. Dudarev, and D. B. Rogozkin, "Coherent backscattering enhancement under conditions of weak wave localisation in disordered 3D and 2D systems," Sov. Phys. JETP **69**, 481–490 (1989).
23. M. I. Mishchenko, "Polarized light transfer in a medium containing perfectly oriented elongated particles," Kinem. Phys. Celest. Bodies **3**, 53–62 (1987).
24. M. I. Mishchenko and E. G. Yanovitskij, "Polarized light transfer in a medium containing perfectly oriented elongated particles. General theory. 'Rayleigh' scattering," Kinem. Phys. Celest. Bodies **4**, 19–29 (1988).
25. M. I. Mishchenko, J. M. Dlugach, and E. G. Yanovitskij, "Multiple light scattering by polydispersions of randomly distributed perfectly-aligned, infinite Mie cylinders illuminated perpendicularly to their axes," J. Quantum Spectrosc. Radiat. Transfer **47**, 401–410 (1992).
26. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Nonrelativistic Theory* (Pergamon, New York, 1977).
27. V. M. Galitzkiy and V. I. Kogan, *Problems in Quantum Mechanics* (Prentice-Hall, Englewood Cliffs, N.J., 1963).
28. A. P. Prudnikov, Yu. A. Brychkov, and O. U. Marichev, *Integrals and Series, 3, More Special Functions* (Gordon and Breach, New York, 1986–1992).
29. M. Abramovitz and I. A. Stegan, *Handbook of Mathematical Functions*, Applied Mathematics Series No. 55 (U.S. National Bureau of Standards, Washington, D.C., 1964).